ON THE BEHAVIOUR OF STANLEY DEPTH UNDER VARIABLE ADJUNCTION

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Abstract. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over the field K. For integers $1 \leq t < n$ consider the ideal $I = (x_1, \ldots, x_t) \cap (x_{t+1}, \ldots, x_n)$ in S. In this paper we bound from above the Stanley depth of the ideal $I' = (I, x_{n+1}, \ldots, x_{n+p}) \subset S' = S[x_{n+1}, \ldots, x_{n+p}]$. We give similar upper bounds for the Stanley depth of the ideal $(I_{n,2}, x_{n+1}, \ldots, x_{n+p})$, where $I_{n,2}$ is the squarefree Veronese ideal of degree 2 in n variables.

1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K and M be a finitely generated $\mathbb{Z}^n$-graded S-module. If $u \in M$ is a homogeneous element in M and $Z \subset \{x_1, \ldots, x_n\}$ then let $uK[Z] \subset M$ denote the linear K-subspace of all elements of the form uf, $f \in K[Z]$. This space is called a Stanley space of dimension $|Z|$ if $uK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of module M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D} : M = \bigoplus_{i=1}^r u_i K[Z_i]$. Set $\operatorname{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of M. Stanley depth is an invariant which has some common properties with the homological depth invariant.

In 1982 Stanley conjectured (see [12]) that $\operatorname{sdepth} M \geq \operatorname{depth} M$. This conjecture is still open except some results obtained mainly for $n \leq 5$ (see [1], [2], [9], [10], [11]). A method to compute the Stanley depth is given in [6]. Even when it does not provide the value of the Stanley depth, this method allows one to obtain fairly good estimations for the invariant of interest.

In [7], Ishaq proved that if J is a monomial ideal of $S = K[x_1, \ldots, x_n]$ and $J' = (J, x_{n+1})$ is the ideal of $S' = S[x_{n+1}]$ then $\operatorname{sdepth}(J) \leq \operatorname{sdepth}(J') \leq \operatorname{sdepth}(J) + 1$. When adjoining more variables, a similar result can be easily obtained by iterating Ishak's result. However, the upper bound for $\operatorname{sdepth}((J, x_{n+1}, \ldots, x_{n+p}))$ $(p \geq 2)$ thus obtained can be sometimes too pessimistic.

The aim of this paper is to bound from above the Stanley depth of ideals obtained by adjoining variables to monomial ideals in S belonging to two classes. A first class consists of radical monomial ideals described in the theorem below, whose proof is given in the next section.

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Theorem 2.1. *Let $I = (x_1, \dots, x_t) \bigcap (x_{t+1}, \dots, x_n)$ be a monomial ideal in $S = K[x_1, \dots, x_n]$, where $1 \le t < n$, and let $I' = (I, x_{n+1}, \dots, x_{n+p}) \subset S' = S[x_{n+1}, \dots, x_{n+p}]$, where $p \ge 2$. Then*

$$\operatorname{sdepth}(I') \le 2 + \frac{\binom{n}{3} - \binom{t}{3} - \binom{n-t}{3} + p\binom{n}{2} + n\binom{p}{2} + \binom{p}{3}}{t(n-t) + np - \frac{p(n+2)}{4}}. \tag{1.1}$$

An alternative bound is obtained by imposing some conditions on t, n and p, see Theorem 2.3 in Section 2 for the precise statement.

The reasoning used to prove the results mentioned above can be adapted to work for another class of ideals, namely, squarefree Veronese ideals of degree 2. In Section 3 we shall prove the following.

Theorem 3.1. *Let $I_{n,2}$ be the squarefree Veronese ideal of degree 2 in S and $(I_{n,2}, x_{n+1}, \dots, x_{n+p})$ be the squarefree ideal in S', where $p \ge 2$. Then*

$$\operatorname{sdepth}(I_{n,2}, x_{n+1}, \dots, x_{n+p}) \le 2 + \frac{\binom{n+p}{3}}{\binom{n}{2} + np - p - \frac{p}{2}\lfloor \binom{n}{3} / \binom{n}{2} \rfloor}.$$

Also this bound is further improved by imposing some condition on n and p (cf. Theorem 3.4).

Herzog, Vlădoiu and Zhang [6] have results implying that Stanley's conjecture is true for squarefree Veronese ideals. In Section 3 we note that Stanley's conjecture is valid for the ideal obtained by adding several variables to a squarefree Veronese ideal.

Proposition 3.8. *Let $I \subset S = K[x_1, \dots, x_n]$ be the squarefree Veronese ideal generated by all monomials of degree d and $I' = (I, x_{n+1}, \dots, x_{n+m}) \subset S' = S[x_{n+1}, \dots, x_{n+m}]$. Then Stanley's conjecture holds for the ideal I'.*

In the last section of the paper we compare the bounds which we obtained without conditions with those which we obtained when appropriate conditions are imposed.

Some results from [4], [5], [7] and [8] are very important for our estimations of Stanley depth and precise references will be given in appropriate places. For unexplained notation, the reader is referred to [6].

2. Upper bounds for the Stanley depth of squarefree monomial ideal when some variables are added

Theorem 2.1. *Let $I = (x_1, \dots, x_t) \bigcap (x_{t+1}, \dots, x_n)$ be a monomial ideal in $S = K[x_1, \dots, x_n]$, where $1 \le t < n$, and let $I' = (I, x_{n+1}, \dots, x_{n+p}) \subset S' =$*

$S[x_{n+1},\dots,x_{n+p}]$, *where* $p\geq 2$. *Then*

$$\operatorname{sdepth}(I')\leq 2+\frac{\binom{n}{3}-\binom{t}{3}-\binom{n-t}{3}+p\binom{n}{2}+n\binom{p}{2}+\binom{p}{3}}{t(n-t)+np-\frac{p(n+2)}{4}}.$$

Proof. Note that I' is a squarefree monomial ideal generated by monomials of degree 2 and 1. Let $k=\operatorname{sdepth}(I')$. The poset $P_{I'}$ has the partition $\mathcal{P}: P_{I'}=\bigcup_{i=1}^{s}[C_i,D_i]$, satisfying $\operatorname{sdepth}(\mathcal{D}(\mathcal{P}))=k$, where $\mathcal{D}(\mathcal{P})$ is the Stanley decomposition of I' with respect to the partition $\mathcal{P}$. We may choose $\mathcal{P}$ such that $|D|=k$ whenever $C\neq D$ in the interval $[C,D]$.

For each interval $[C_i,D_i]$ in $\mathcal{P}$ with $|C_i|=2$ when in the corresponding monomial, one variable belongs to $\{x_1,\dots,x_t\}$ and one to $\{x_{t+1},\dots,x_n\}$ we have $|D_i|-|C_i|$ subsets of cardinality 3 in this interval. Now for each interval $[C_j,D_j]$ when $|C_j|=1$ we have at least $\binom{k-1}{2}$ subsets of cardinality 3 in this interval. We have p such intervals. So we have $p\binom{k-1}{2}$ subsets of cardinality 3.

Now we consider those intervals $[C_l,D_l]$ such that $|C_l|=2$ and the corresponding monomial is of the form x_lx_λ, where $x_l\in\{x_{n+1},\dots,x_{n+p}\}$. Now either $x_\lambda\in\{x_1,\dots,x_n\}$ or $x_\lambda\in\{x_{n+1},\dots,x_{n+p}\}$. If $x_\lambda\in\{x_1,\dots,x_n\}$ then we have np such intervals and each has at least $k-2$ subsets of cardinality 3. If $x_\lambda\in\{x_{n+1},\dots,x_{n+p}\}$ then we have $\binom{p}{2}$ such intervals and each has at least $k-2$ subsets of cardinality 3. Some subsets of cardinality 2 of the form C_l already appear in the intervals $[C_j,D_j]$ and such subsets are $p(k-1)$ in number. Since the partition is disjoint, we subtract this from total number of C_l's, so that we have at least

$$\Big[\binom{n}{2}-\binom{t}{2}-\binom{n-t}{2}\Big](k-2)+p\binom{k-1}{2}+\Big[np+\binom{p}{2}-p(k-1)\Big](k-2)$$

subsets of cardinality 3. This number is less than or equal to the total number of subsets of cardinality 3. So

$$\begin{aligned}&\Big[\binom{n}{2}-\binom{t}{2}-\binom{n-t}{2}\Big](k-2)+\Big[np+\binom{p}{2}-\frac{p(k-1)}{2}\Big](k-2)\\ &\qquad\leq\binom{n}{3}-\binom{t}{3}-\binom{n-t}{3}+p\binom{n}{2}+n\binom{p}{2}+\binom{p}{3}.\end{aligned}\tag{2.1}$$

Now we know by [7, Theorem 2.11] that $k\leq\frac{n+2}{2}+p$. This implies $-(k-1)\geq-\frac{n+2}{2}-p+1$. Using this in the left side of inequality (2.1), one gets

$$\begin{aligned}&\Big[t(n-t)+np-\frac{p(n+2)}{4}\Big](k-2)\\ &\leq\Big[\binom{n}{2}-\binom{t}{2}-\binom{n-t}{2}\Big](k-2)+\Big[np+\binom{p}{2}-\frac{p(k-1)}{2}\Big](k-2).\end{aligned}$$

Combining both inequalities we get the required result. $\square$

Example 2.2. *Let us consider* $I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \subseteq S = K[x_1, \ldots, x_6]$. *By* [7, Theorem 2.8], *we have* $\operatorname{sdepth}(I) \leq 4$. *Let* $I' = (I, x_7, x_8, x_9) \subseteq S' = S[x_7, x_8, x_9]$ *then by* [7, Lemma 2.11] *we have* $\operatorname{sdepth}(I') \leq 7$. *Now by our Theorem 2.1 we have* $\operatorname{sdepth}(I') \leq 5$.

We can further improve the upper bound if we impose some additional condition on n, t and p.

Formula (2.1) in the proof of Theorem 2.1 is equivalent to

$$\begin{aligned}0 \leq 3pk^2 - 3(2np + 2nt + p^2 - 2t^2 + 2p)k \\ + 6np + 6nt + 3n^2p - 6t^2 - 3nt^2 + 3n^2t + 3p^2 + 3np^2 + 2p + p^3.\end{aligned}$$

Consider it as a quadratic polynomial in k of discriminant

$$D := (36pt + 36t^2)n^2 - 36(t^2p - tp^2 + 2t^3)n + 12p^2 - 36p^2t^2 - 3p^4 + 36t^4. \tag{2.2}$$

Since this quadratic polynomial in n has the discriminant

$$\Delta := 432tp^2(t + p)(3t^2 + 3pt - 4 + p^2)$$

obviously positive, we have $D \geq 0$ for either

$$n \leq t - \frac{p}{2} - p\sqrt{1 + \frac{p^2 - 4}{3t(t + p)}}$$

or

$$n \geq t - \frac{p}{2} + p\sqrt{1 + \frac{p^2 - 4}{3t(t + p)}}. \tag{2.3}$$

The former possibility is excluded by the fact that $n > t$, so that, assuming the latter inequality, we conclude that either

$$k \leq n + \frac{p}{2} + \frac{t(n - t)}{p} + 1 - \frac{\sqrt{D}}{6p}$$

or

$$k \geq n + \frac{p}{2} + \frac{t(n - t)}{p} + 1 + \frac{\sqrt{D}}{6p}. \tag{2.4}$$

The latter bound for k does not hold (see Lemma 2.4 below). We have thus obtained the following result.

Theorem 2.3. *Keep the notation and hypotheses of Theorem 2.1. If*

$$n \geq t - \frac{p}{2} + p\sqrt{1 + \frac{p^2 - 4}{3t(t + p)}}$$

then

$$\operatorname{sdepth}(I') \leq n + \frac{p}{2} + \frac{t(n - t)}{p} + 1 - \frac{\sqrt{D}}{6p}, \tag{2.5}$$

where

$$D = (36pt + 36t^2)n^2 - 36(t^2p - tp^2 + 2t^3)n + 12p^2 - 36p^2t^2 - 3p^4 + 36t^4.$$

Lemma 2.4. *Conditions* (2.3) *and* (2.4) *do not hold simultaneously.*

Proof. Suppose that both inequalities (2.3) and (2.4) are satisfied. From the relation $k \leq p + 1 + n/2$ known from [7, Theorem 2.11] it results, on the one hand, that $p > n$ and, on the other hand, that $p(p - n) > 2t(n - t)$, so that

$$p^2 + 2t^2 > (p + 2t)n. \tag{2.6}$$

From (2.3) we obtain in particular

$$n > t + \frac{p}{2}.$$

This and (2.6) give

$$p > 4t. \tag{2.7}$$

We shall discuss two cases.

Case $t \geq 2$. It is easily seen that the function $p \mapsto \frac{p^2-4}{3t(t+p)}$ is increasing. Therefore

$$\frac{p^2 - 4}{3t(t + p)} > \frac{16t^2 - 4}{15t^2} \geq 1.$$

From (2.3) it then follows $n > t + p(\sqrt{2} - 0.5) > t + 0.91p$. Using this in (2.6), we get $0.09p > 2.82t$, whence $p > 31t$. Then

$$\frac{p^2 - 4}{3t(t + p)} > \frac{961t^2 - 4}{96t^2} \geq 10,$$

so that $n > t + (\sqrt{11} - 0.5)p > p$. This is a contradiction, which shows that our assumption is false in this case.

Case $t = 1$. From $p > 4$ we now get

$$\frac{p^2 - 4}{3(1 + p)} \geq \frac{7}{6}$$

and $n \geq 1 + \left(\sqrt{\frac{13}{6}} - 0.5\right) p > 0.97p + 1$. Using this lower bound for n in (2.6), we get $0.03p > 2.94$, and therefore $p > 98 > 31t$. We have seen that this contradicts $p > n$. □

Example 2.5. *For $n = 7$, $t = 3$, $p = 5$, the latter theorem gives $k \leq 7$, while Theorem 2.1 yields a slightly weaker bound $k \leq 8$. However, for $n = 66$, $t = 2$, $p = 3$ one gets $k \leq 42$ by using Theorem 2.3 and $k \leq 41$ when applying Theorem 2.1.*

Corollary 2.6. *Let $I = Q \bigcap Q'$ be a monomial ideal in $S = K[x_1, \ldots, x_n]$ where Q and Q' are monomial primary ideals in S such that $\sqrt{Q} = (x_1, \ldots, x_t)$ and $\sqrt{Q'} = (x_{r+1}, \ldots, x_n)$ for some integers $1 \leq r \leq t < n$. Then*

$$\operatorname{sdepth}(I) \leq 2+$$

$$\frac{\binom{n-t+r}{3} - \binom{r}{3} - \binom{n-t}{3} + (t-r)\binom{n-t+r}{2} + (n-t+r)\binom{t-r}{2} + \binom{t-r}{3}}{r(n-t) + (n-t+r)(t-r) - \frac{(t-r)(n-t+r+2)}{4}}.$$

Proof. Note that $\sqrt{I} = (P' \cap S', x_{r+1}, \dots, x_t)$ where $S' = K[x_1, \dots, x_r, x_{t+1}, \dots, x_n]$ and $P' = (x_1, \dots, x_r) \cap (x_{t+1}, \dots, x_n) \subset S'$. Now we can apply Theorem 2.1 and [7, Theorem 2.1]. $\square$

Example 2.7. *Let $I = Q \cap Q'$ be a monomial ideal in $S = K[x_1, \dots, x_8]$, where Q and Q' are monomial primary ideals with $\sqrt{Q} = (x_1, \dots, x_6)$ and $\sqrt{Q'} = (x_5, \dots, x_8)$. Then by* [7, Proposition 2.13] *we have* $\operatorname{sdepth} I \leq 6$ *and by our Corollary 2.6 we have* $\operatorname{sdepth}(I) \leq 5$.

3. Upper bounds for the Stanley depth of squarefree Veronese ideal when some variables are added

We denote by $I_{n,d}$ the squarefree Veronese ideal of degree d in the polynomial ring in n variables over a field K. Our first bound for the Stanley depth of such an ideal is given by the next result.

Theorem 3.1. *Let K be a field and n, $p \geq 2$ integers. Let $I_{n,2}$ be the squarefree Veronese ideal in $S = K[x_1, \dots, x_n]$ and $I' = (I_{n,2}, x_{n+1}, \dots, x_{n+p}) \subseteq S' = S[x_{n+1}, \dots, x_{n+p}]$. Then*

$$\operatorname{sdepth}(I') \leq 2 + \frac{\binom{n+p}{3}}{\binom{n}{2} + np - p - \frac{p}{2}\lfloor \binom{n}{3} / \binom{n}{2} \rfloor}.$$

Proof. Note that I' is a squarefree monomial ideal generated by monomials of degree 2 and 1. Let $k = \operatorname{sdepth}(I')$. The poset $P_{I'}$ has the partition $\mathcal{P} : P_{I'} = \bigcup_{i=1}^{s} [C_i, D_i]$ satisfying $\operatorname{sdepth}(\mathcal{D}(\mathcal{P})) = k$, where $\mathcal{D}(\mathcal{P})$ is the Stanley decomposition of I' with respect to the partition $\mathcal{P}$. We may choose $\mathcal{P}$ such that $|D| = k$ whenever $C \neq D$ in the interval $[C, D]$.

For each interval $[C_i, D_i]$ in $\mathcal{P}$ with $|C_i| = 2$, when in the corresponding monomial both variables belong to $\{x_1, \dots, x_n\}$ we have at least $|D_i| - |C_i|$ subsets of cardinality 3 in this interval. Now for each interval $[C_j, D_j]$, when $|C_j| = 1$ we have at least $\binom{k-1}{2}$ subsets of cardinality 3 and we have p such intervals.

Now we consider those intervals $[C_l, D_l]$ such that $|C_l| = 2$ and the corresponding monomial is of the form $x_l x_\lambda$, where $x_l \in \{x_{n+1}, \dots, x_{n+p}\}$. Now either $x_\lambda \in \{x_1, \dots, x_n\}$ or $x_\lambda \in \{x_{n+1}, \dots, x_{n+p}\}$. If $x_\lambda \in \{x_1, \dots, x_n\}$, then we have np such intervals and each of them has at least $k-2$ subsets of cardinality 3. If $x_\lambda \in \{x_{n+1}, \dots, x_{n+p}\}$ then we have $\binom{p}{2}$ such intervals, each of which having at least $k-2$ subsets of cardinality 3. Some subsets of cardinality 2 of the form C_l already appear in the interval when the interval starts from a single variable, and there are $p(k-1)$ such subsets. Since the partition is disjoint, we subtract this from the total number of C_l's, so that we have at least

$$\binom{n}{2}(k-2) + p\binom{k-1}{2} + \left[np + \binom{p}{2} - p(k-1)\right](k-2)$$

subsets of cardinality 3, and this number is less than or equal to the total number of subsets of cardinality 3. So

$$(3.1)\quad \binom{n}{2}(k-2)+p\binom{k-1}{2}+\Big[np+\binom{p}{2}-p(k-1)\Big](k-2) \leq \binom{n+p}{3}.$$

Now

$$(3.2)\quad \begin{aligned} &\binom{n}{2}(k-2)+p\binom{k-1}{2}+\Big[np+\binom{p}{2}-p(k-1)\Big](k-2) \\ &\qquad = \Big[\binom{n}{2}+np+\binom{p}{2}+\frac{p}{2}(1-k)\Big](k-2). \end{aligned}$$

Since by [5, Theorem 1.2] we know that $\operatorname{sdepth}(I_{n,2}) \leq \lfloor \binom{n}{3}/\binom{n}{2} \rfloor + 2$, applying [7, Theorem 2.11] we get $k \leq \lfloor \binom{n}{3}/\binom{n}{2} \rfloor + 2 + p$.

Putting

$$-k \geq -\lfloor \binom{n}{3}/\binom{n}{2} \rfloor - 2 - p$$

in (3.2), we get

$$\begin{aligned} &\binom{n}{2}(k-2)+p\binom{k-1}{2}+\Big[np+\binom{p}{2}-p(k-1)\Big](k-2) \\ \geq &\Big[\binom{n}{2}+np+\binom{p}{2}+\frac{p}{2}\Big(1-\lfloor \binom{n}{3}/\binom{n}{2} \rfloor - 2 - p\Big)\Big](k-2). \end{aligned}$$

The required result is obtained by combining the above inequality with (3.1). $\square$

Example 3.2. *Let $S = K[x_1, \ldots, x_5]$ and $I_{5,2}$ be the squarefree Veronese ideal. Then by* [4, Corollary 1.5] *or* [5, Theorem 1.2] *we have* $\operatorname{sdepth}(I_{5,2}) = 3$.

Now let $I' = (I_{5,2}, x_6, x_7)$ be the monomial ideal in $S' = S[x_6, x_7]$. By [7, Lemma 2.11] *we have* $\operatorname{sdepth}(I') \leq 5$*, while our Theorem 3.1 yields* $\operatorname{sdepth}(I') \leq 4$.

Example 3.3. *Let $S = K[x_1, \ldots, x_{11}]$ and $I_{11,2}$ be the squarefree Veronese ideal. Then by* [5, Theorem 1.2] *we have* $4 \leq \operatorname{sdepth}(I_{11,2}) \leq 5$.

Let $I' = (I_{11,2}, x_{12}, \ldots, x_{17})$ be the monomial ideal in $S' = S[x_{12}, \ldots, x_{17}]$, then by [7, Lemma 2.11] *we have* $\operatorname{sdepth}(I') \leq 11$ *and by Theorem 3.1* $\operatorname{sdepth}(I') \leq 8$.

If we impose some condition on n and p we can improve the bound given in Theorem 3.1.

The last expression given in the proof of Theorem 3.1 is equivalent to

$$0 \leq 3pk^2 - 3(n^2-n+2np+p^2+2p)k + n^3+3n^2p+3n^2+3np^2+6np-4n+p^3+3p^2+2p.$$

The quadratic in k has discriminant

$$E := 9n^4+(24p-18)n^3+(18p^2-36p+9)n^2-(18p^2-12p)n+12p^2-3p^4$$

obviously positive for $n \geq p$. A simple computation convince ourselves that the discriminant is actually positive for $n \geq p-1$. Since, on the one hand, one has

$E > 9(p-1)^4$ for $p \geq 2$, $n \geq \max\{2, p-1\}$, and, on the other hand, from [5, Theorem 1.2] and [7, Theorem 2.11] it is known that $k \leq p + 2 + \lfloor (n-2)/3 \rfloor$, we conclude that the next result holds.

Theorem 3.4. *Keep the notation and hypotheses from Theorem 3.1. Then for $n \geq p-1$ one has*

$$k \leq \frac{n(n-1)}{2p} + \frac{p}{2} + n + 1 - \frac{\sqrt{E}}{6p},$$

where

$$E = 9n^4 + (24p-18)n^3 + (18p^2 - 36p + 9)n^2 - (18p^2 - 12p)n + 12p^2 - 3p^4.$$

Corollary 3.5. *Let $S' = K[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+p}]$ be a polynomial ring and let $P_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $i = 1, \ldots, n$, be monomial prime ideals in S'. Denote $Q_i = (P_i, x_{n+1}, \ldots, x_{n+p})$. If $\operatorname{Ass}(S'/I') = \{Q_1, \ldots, Q_n\}$, then*

$$\operatorname{sdepth}(I) \leq 2 + \frac{\binom{n+p}{3}}{\binom{n}{2} + np - p - \frac{p}{2}\lfloor \binom{n}{3} / \binom{n}{2} \rfloor}.$$

Example 3.6. *For $n = 11$, $p = 6$, Theorem 3.4 gives $k \leq 7$ instead of $k \leq 8$ cf. Theorem 3.1.*

Example 3.7. *For $n = 5$, $p = 2$, this result gives $k \leq 3$, while Theorem 3.1 yields a slightly weaker bound $k \leq 4$. Therefore, in the situation described in Example 3.2 one has*

$$\operatorname{sdepth}_S(I) = \operatorname{sdepth}_{S'}(I').$$

We now prove that Stanley's conjecture is verified by ideals of the type studied in this section.

Proposition 3.8. *For positive integers n and d, let $I \subset S = K[x_1, \ldots, x_n]$ be the squarefree Veronese ideal generated by all monomials of degree d and $I' = (I, x_{n+1}, \ldots, x_{n+m}) \subset S' = S[x_{n+1}, \ldots, x_{n+m}]$. Then Stanley's conjecture holds for the ideal I'.*

Proof. From $\operatorname{depth}_{S'}(S'/I') = \operatorname{depth}_S(S/I)$ it follows $\operatorname{depth}_{S'}(I') = \operatorname{depth}_S(I)$. As a consequence of results established in [6] (or by applying [4, Corollary 1.2]), Stanley's conjecture holds for squarefree Veronese ideals, so that $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$. By [8, Lemma 2.1], the sdepth does not decrease when passing from I to I'. Therefore, Stanley's conjecture holds for I', too. $\square$

4. Comparison of bounds

First we compare the bounds provided in Theorems 3.1 and 3.4. The outcome of our study is the following.

Theorem 4.1. *Let K be a field and $n,\ p \geq 2$ integers. Let $I_{n,2}$ be the squarefree Veronese ideal in $S = K[x_1, \dots, x_n]$ and $I' = (I_{n,2}, x_{n+1}, \dots, x_{n+p}) \subseteq S' = S[x_{n+1}, \dots, x_{n+p}]$. If $n \geq p-1$ then the bound for* $\operatorname{sdepth}(I')$ *given by Theorem 3.4 is smaller than that given by Theorem 3.1.*

Since

$$\left\lfloor \binom{n}{3} / \binom{n}{2} \right\rfloor = \lfloor (n-2)/3 \rfloor,$$

we shall distinguish the values of n according to their residues mod 3.

Case $n = 3s+1,\ s \geq 1$**.** The bound given in Theorem 3.1 specialises to

$$u_1 := \frac{27s^3 + 27s^2(p+2) + 3s(3p^2+10p+5) + p^3 + 5p}{3s(9s+5p+3)+3p}, \tag{4.1}$$

while that given in Theorem 3.4 becomes in this case

$$l_1 := \frac{27s^2 + 9(2p+1)s + 3p^2 + 12p - \sqrt{dv_1}}{6p}, \tag{4.2}$$

with

$$dv_1 := 729s^4 + (648p+486)s^3 + (162p^2+324p+81)s^2 + (54p^2+36p)s + 12p^2 - 3p^4.$$

We want to know for what values of s we have $l_1 \leq u_1$ for all $p \geq 2$, or equivalently

$$\left(9s^2 + (5p+3)s + p\right)\sqrt{dv_1} \geq r_1,$$

where

$$r_1 := 243s^4 + (243p+162)s^3 + (63p^2+126p+27)s^2 + (15p+27p^2-3p^3)s + 2p^2 + 3p^3 - 2p^4.$$

The second derivative of the function function $r_1 : [(p-2)/3, +\infty) \longrightarrow \mathbb{R}$ being positive, the first derivative is at least as large as

$$r_1'(\frac{p-2}{3}) = 3(52p^3 - 153p^2 + 135p - 36).$$

Since the expression in the right side is positive for $p \geq 2$, for these values r_1 is greater than or equal to

$$r_1(\frac{p-2}{3}) = 2(2p-3)(p-2)(2p-1)^2,$$

which is nonnegative for $p \geq 2$. This analysis shows that the desired inequality is equivalent to that obtained by squaring it, which, with some computer assistance, is found to be

$$p^2(3s+1+p)(3s+p)(3s-1+p)\left(s^2(15p-18) + (6p^2-3p-6)s - p^3 + 3p^2 - 2p\right) \geq 0.$$

This is true if and only if

$$f_1 := (15p-18)s^2 + (6p^2-3p-6)s - p^3 + 3p^2 - 2p \geq 0 \quad \text{for } p \geq 2, s \geq \max\{2, (p-2)/3\}.$$

Since the discriminant

$$D_1 = 96p^4 - 288p^3 + 273p^2 - 108p + 36$$

is positive for $p \geq 2$, $f_1(s) \geq 0$ if and only if

$$s \geq \frac{-(6p^2 - 3p - 6) + \sqrt{D_1}}{6(5p - 6)} =: s_1.$$

In terms of the number of variables n, this means that the bound given in Theorem 3.4 is better than that given in Theorem 3.1 for

$$n \geq n_1 := \frac{\sqrt{D_1} - 6p^2 + 13p - 6}{2(5p - 6)}.$$

Case $n = 3s + 2$, $s \geq 0$. Now

$$\begin{aligned}
u_2 &:= \frac{27s^3 + 27(p+3)s^2 + (9p^2 + 48p + 60)s + p^3 + 3p^2 + 14p + 12}{27s^2 + 3(5p+9)s + 6p + 6}, \\
l_2 &:= \frac{27s^2 + 9(2p+3)s + 3p^2 + 18p + 6 - \sqrt{dv2}}{6p}, \\
dv_2 &:= 729s^4 + (648p + 1458)s^3 + (162p^2 + 972p + 1053)s^2 \\
&\qquad + (162p^2 + 468p + 324)s - 3p^4 + 48p^2 + 72p + 36.
\end{aligned}$$

Since dv_2 increases with s, its minimal value in the range of interest is

$$dv_2(\frac{p-3}{3}) = 3(p-2)(16p^3 - 40p^2 + 27p - 6) \geq 0.$$

Therefore, $l_2 \leq u_2$ is equivalent to

$$\left(9s^2 + (5p+9)s + 2p + 2\right)\sqrt{dv_2} \geq r_2,$$

with

$$\begin{aligned}
r_2 &:= 243s^4 + 243(p+2)s^3 + (63p^2 + 351p + 351)s^2 \\
&\qquad + (-3p^3 + 57p^2 + 162p + 108)s - 2p^4 + 14p^2 + 24p + 12.
\end{aligned}$$

We further find

$$r_2' \geq r_2'\left(\frac{p-3}{3}\right) = 3(52p^3 - 161p^2 + 141p - 36) > 0 \quad \text{for } p \geq 2,$$

so that

$$r_2 \geq r_2\left(\frac{p-3}{3}\right) = 2(8p^4 - 40p^3 + 61p^2 - 33p + 6) > 0 \quad \text{for } p \geq 3.$$

As for $p = 2$ the right side of the desired inequality $l_2 \leq u_2$ is $3(3s+2)(27s^3+90s^2+85s+14) > 0$, we may square both sides of the inequality under study and find that for $p \geq 2$ it is equivalent to

$$p^3(p+2+3s)(p+1+3s)(p+3s)\left(15s^2 + (6p+15)s - p^2 + 3p + 4\right) \geq 0.$$

This holds precisely when

$$f_2 := 15s^2 + (6p+15)s - p^2 + 3p + 4 \geq 0.$$

The discriminant being $96p^2 - 15 > 0$, f_2 takes positive values for

$$s \geq \frac{-3(2p+5) + \sqrt{96p^2 - 15}}{30} =: s_2.$$

Thus we conclude that $l_2 \leq u_2$ holds for $n \equiv 2 \pmod 3$ and

$$n \geq n_2 := \frac{\sqrt{96p^2 - 15} - 6p + 5}{10}.$$

Case $n = 3s$, $s \geq 1$. We study the inequality $l_3 \leq u_3$, with

$$l_3 := \frac{27s^2 + 9(2p-1)s + 3p^2 + 6p - \sqrt{dv_3}}{6p},$$
$$u_3 := \frac{27s^3 + 27(p+1)s^2 + (9p^2 + 12p - 12)s + p^3 - 3p^2 - 4p}{27s^2 + 3(5p-3)s - 3p},$$

and

$$dv_3 := 729s^4 + (648p - 486)s^3 + (162p^2 - 324p + 81)s^2 + (-54p^2 + 36p)s + 12p^2 - 3p^4.$$

With arguments similar to those given in the previous cases one finds

$$dv_3 \geq dv_3(\frac{p-1}{3}) = 3(p-2)(16p^3 - 40p^2 + 27p - 6) \geq 0 \quad \text{for } p \geq 2.$$

Therefore, $l_3 \leq u_3$ is equivalent to

$$\left(9s^2 + (5p-3)s - p\right)\sqrt{dv_3} \geq r_3,$$

where

$$r_3 := 243s^4 + (243p - 162)s^3 + (63p^2 - 126p + 27)s^2 + (-3p^3 - 21p^2 + 15p)s + 2p^2 + 3p^3 - 2p^4.$$

After we check that r_3 is positive in the range $p \geq 2$, $s \geq \max\{1, (p-1)/3\}$, we may square the last inequality and find that it is equivalent to

$$4p^2(3s+p-2)(3s+p)(3s+p-1)\left(s^2(15p-9) + (6p^2 - 9p + 3)s - p^3 + p\right) \geq 0.$$

Since the quadratic polynomial in s

$$f_3 = (15p-9)s^2 + (6p^2 - 9p + 3)s - p^3 + p$$

has discriminant

$$D_3 := 3(p-1)(32p^3 - 16p^2 + 3p - 3) > 0,$$

we have $f_3(s) \geq 0$ if and only if

$$s \geq \frac{-6p^2 + 9p - 3 + \sqrt{D_3}}{6(5p-3)} =: s_3.$$

The conclusion is that, for $n \equiv 3 \pmod 3$, the bound provided in Theorem 3.4 is tighter than that given in Theorem 3.1 if and only if

$$n \geq \frac{-6p^2 + 9p - 3 + \sqrt{D_3}}{2(5p-3)} =: n_3.$$

It remains to compare n_1, n_2, n_3 and $p - 1$.

Lemma 4.2. *One has $n_1 = 1$, $n_2 \simeq 1.22$, $n_3 \simeq 1.07$ for $p = 2$, and $p - 1 \geq n_2 > n_3 > n_1$ for $p \geq 3$.*

Proof. Assume $p \geq 3$. The inequality $n_2 > n_3$ is successively equivalent to

$$(5p-3)\sqrt{96p^2 - 15} > 2p + 5\sqrt{D_3},$$

$$36p^3 - 47p^2 + 45p - 18 > p\sqrt{D_3},$$

and

$$(5p-3)^2(12p^4 - 18p^3 + 28p^2 - 15p + 9) > 0,$$

which is obviously true.

The inequality $n_1 < n_3$ is rewritten

$$2p^2 + (5p-3)\sqrt{D_1} < (5p-6)\sqrt{D_3}.$$

Squaring this, one finds after some easy computations

$$(5p-3)p\sqrt{D_1} < (5p-3)(36p^3 - 119p^2 + 150p - 72).$$

After simplification and squaring one gets

$$1200p^6 - 8424p^5 + 24904p^4 - 40866p^3 + 39627p^2 - 21600p + 5184 > 0,$$

which is readily checked to be true for $p \geq 3$.

Finally, $n_2 < p - 1$ is put into the equivalent form

$$\sqrt{96p^2 - 15} < 16p - 5p,$$

which holds because the left side is less than $10p$, while the right side is at least $11p$ for $p \geq 3$. $\square$

Now the proof of Theorem 4.1 is complete.

The bounds for the class of ideals studied in Section 2 can be compared by analogue reasoning. The details of the analysis are, however, much more involved. As seen by Example 2.5, none of Theorems 2.1 and 2.3 is uniformly better than the other. Our final result specifies conditions under which Theorem 2.3 yields a tighter bound than that given in Theorem 2.1.

Theorem 4.3. *Let $I = (x_1, \ldots, x_t) \bigcap (x_{t+1}, \ldots, x_n)$ be a monomial ideal in $S = K[x_1, \ldots, x_n]$, where $1 \leq t < n$, and let $I' = (I, x_{n+1}, \ldots, x_{n+p}) \subset S' = S[x_{n+1}, \ldots, x_{n+p}]$, where $p \geq 2$. Suppose that it holds*

$$n \geq n_0 := t - \frac{p}{2} + p\sqrt{1 + \frac{p^2 - 4}{3t(t+p)}}.$$

Then the bound for $\operatorname{sdepth}(I')$ *given in* (2.5) *is tighter than that given in* (1.1) *if and only if*

$$0 \leq 3n^2 + 6np - 4p^2 + 4 \quad \textit{and} \quad \max\{1, t_l\} \leq t \leq \min\{n-1, t_u\},$$

where

$$t_l := \frac{6n - \sqrt{6(3n^2 + 6np - 4p^2 + 4)}}{12}, \quad t_u := \frac{6n + \sqrt{6(3n^2 + 6np - 4p^2 + 4)}}{12}.$$

Proof. For $n = 2$ one has $t = 1$ and therefore (by hypothesis $n \geq n_0$) $p = 2$, so that the bounds given in Theorems 2.1 and 2.3 are 10/3 and 3, respectively. From now on we shall assume $n \geq 3$.

With notation

$$L := n + \frac{p}{2} + \frac{t(n-t)}{p} + 1 - \frac{\sqrt{D}}{6p},$$

$$U := 2 + \frac{\binom{n}{3} - \binom{t}{3} - \binom{n-t}{3} + p\binom{n}{2} + n\binom{p}{2} + \binom{p}{3}}{t(n-t) + np - \frac{p(n+2)}{4}},$$

we have to find when does it hold $L \leq U$. Routine computations bring this inequality to the equivalent form

$$f_4 \leq g_4\sqrt{D}, \tag{4.3}$$

with

$$\begin{aligned} f_4 &:= (6p^2 + 30tp + 24t^2)n^2 + (-3p^3 + 12p^2t - 6p^2 - 12tp - 30t^2p - 48t^3)n \\ &\qquad - 4p^4 + 6p^3 + 4p^2 - 12p^2t^2 + 12t^2p + 24t^4, \\ g_4 &:= 4nt + 3np - 4t^2 - 2p, \\ D &:= (36pt + 36t^2)n^2 - 36(t^2p - tp^2 + 2t^3)n + 12p^2 - 36p^2t^2 - 3p^4 + 36t^4. \end{aligned}$$

The discriminant of f_4, which is found to be

$$\begin{aligned} df_4 := 3p^2 \big(&35p^4 + (136t - 36)p^3 + (332t^2 - 264t - 20)p^2 \\ &+ (336t^3 - 264t^2 - 112t)p + 108t^4 - 48t^3 - 80t^2\big), \end{aligned}$$

is positive in our hypothesis (for $t \geq 2$ the coefficients of powers of p are obviously positive, and a direct verification leads to the same conclusion if $t = 1$). Therefore, f_4 takes nonnegative values for either

$$n \leq n_l := \frac{3p^3 - 12p^2t + 6p^2 + 12tp + 30t^2p + 48t^3 - \sqrt{df_4}}{2(6p^2 + 30tp + 24t^2)}$$

or

$$n \geq n_u := \frac{3p^3 - 12p^2t + 6p^2 + 12tp + 30t^2p + 48t^3 + \sqrt{df_4}}{2(6p^2 + 30tp + 24t^2)}.$$

As will shall prove in Lemma 4.4 below, one has $n_u \leq n_0$. Therefore, the hypothesis of Theorem 4.3 ensures that Eq. (4.3) is equivalent to

$$h := g_4^2 D - f_4^2 \geq 0..$$

With some computer assistance, we find

$$h = 4p^3 h_1 h_2,$$

where the quadratic polynomials in t

$$\begin{aligned} h_1 &:= 3(n-2)t^2 - (3n^2 - 6n)t + 3n^2p + 3np^2 - 6np + 2p - 3p^2 + p^3, \\ h_2 &:= 24t^2 - 24nt + 3n^2 - 6np + 4p^2 - 4 \end{aligned}$$

have discriminant

$$\Delta_1 := 3(n-2)(3n^3 - 6n^2 + 12n^2p + 12np^2 - 24np + 4p^3 - 12p^2 + 8p)$$

and respectively

$$\Delta_2 := 288n^2 + 576np - 384p^2 + 384.$$

Each discriminant is increasing with n, so that

$$\Delta_1 \geq \Delta_1(3) = 3(4p^3 + 24p^2 + 44p + 27) > 0.$$

Hence, h_1 always has the real roots

$$t_1 := \frac{3n^2 - 6n - \sqrt{\Delta_1}}{6(n-2)}, \quad t_2 := \frac{3n^2 - 6n + \sqrt{\Delta_1}}{6(n-2)},$$

while h_2 has the real roots

$$t_3 := \frac{6n - \sqrt{6(3n^2 + 6np - 4p^2 + 4)}}{12}, \quad t_4 := \frac{6n + \sqrt{6(3n^2 + 6np - 4p^2 + 4)}}{12}$$

provided that

$$0 \leq 3n^2 + 6np - 4p^2 + 4.$$

In Lemma 4.5 below we show that $t_1 < 1$ and $n-1 < t_2$. Therefore, h_1 is negative for all admissible values of t, whence $h \geq 0$ is equivalent to $h_2 \leq 0$. The latter inequality is valid precisely when the conclusion of Theorem 4.3 holds. □

The proof of Theorem 4.3 is complete as soon as we prove the next lemmas.

Lemma 4.4. *One has $n_u \leq n_0$.*

Proof. The desired inequality is successively equivalent to

$$(60tp + 12p^2 + 48t^2)\sqrt{1 + \frac{p^2 - 4}{3t(t+p)}} + 6t^2 - 6tp - 9p^2 - 12t - 6p \geq \sqrt{df_4}$$

and

$$16p^7 + 116p^6 + (776t^2 - 128)p^5 + 4t(763t^2 + 66t - 256)p^4 + (6837t^4 + 876t^3 - 5068t^2 + 256)p^3 + t(8517t^4 + 1884t^3 - 11708t^2 - 1056t + 2240)p^2 + 8t^2(711t^4 + 225t^3 - 1578t^2 - 204t + 712)p + 4t^3(135t^2 + 48t - 244)(3t^2 - 4) \geq 0.$$

For $t = 1$, the last inequality becomes

$$(p+2)(16p^6 + 84p^5 + 480p^4 + 1332p^3 + 237p^2 - 597p + 122) \geq 0,$$

for $t = 2$

$$8(p+4)(2p^3 + 9p^2 + 210p + 392)(p+2)^3 \geq 0,$$

while for $t \geq 3$ all powers of p have positive coefficients. □

Lemma 4.5. *One has $t_1 < 1$, $n - 1 < t_2$.*

Proof. The inequality $t_1 < 1$ is readily brought to the equivalent form

$$0 < (p+1)\big(3n^2 + 3(p-3)n + p^2 - 4p + 6\big),$$

which is obviously true for $n \geq \max\{2, p-1\}$. Since $t_1 + t_2 = n$, we also have $n - 1 < t_2$. □

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